SPACETIMES ADMITTING A 3-PARAMETER SIMILARITY GROUP

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Abstract

Spacetimes admitting a similarity group are considered. Amongst them, special attention is given to the 3-parameter ones. A classification of such spacetimes is given based on the Bianchi type of the similarity group H_3, and the general form of the metric is provided in each case assuming the orbits are non-null.

1 Introduction

Spacetimes admitting (local) groups of (local) isometries [1] have been widely studied, especially in those cases where the dimension of the Lie algebra of Killing vectors fields (KV) is high or where there is a non-trivial isotropy subgroup. On the other hand, Hall and Steele [2] have investigated those spacetimes admitting an r-dimensional Lie algebra of homothetic vector fields (HVF) (which gives rise to an r-parameter group of similarities); solving the problem completely in those cases where $r \geq 6$ and also giving some general results when $r \leq 5$. The purpose of this paper is, up to a certain extent, to complement the study carried out in the above reference, especially in the case $r = 3$, which could be of interest in Cosmology (see, for instance [3, 4, 5] and references cited therein).

Throughout this paper (M, g) will denote a spacetime: M then being a (smooth) Hausdorff, simply connected 4-dimensional manifold, and g a (smooth) Lorentz metric with signature (-+++). A semicolon will denote a covariant derivative with respect to the metric connection associated with g, and a comma will denote a partial derivative as usual. A global vector field X on M is called homothetic if either of the two equivalent conditions

$$\mathcal{L}_X g_{ab} \equiv X_{a;b} + X_{b;a} = 2\lambda g_{ab}$$

$$X_{a;b} = \lambda g_{ab} + F_{ab} \quad (F_{ab} = -F_{ba}) \tag{1}$$

holds on a local chart, where λ is a constant on M, F is the homothetic bivector, and $\mathcal{L}$ denotes the Lie derivative operator. If $\lambda \neq 0$, X is called proper homothetic and it can always be scaled so as to have $\lambda = 1$, if $\lambda = 0$ then X is a KV on M. For a geometrical interpretation of (1) we refer the reader to [6, 7].

A necessary condition that X be homothetic is

$$X^a{}_{;bc} = R^a{}_{bcd} X^d \tag{2}$$

where $R^a{}_{bcd}$ are the components of the Riemann tensor in a coordinate chart; thus, an HVF

is a particular case of affine collineation [8] and therefore it will satisfy

$$\mathcal{L}_X R^a{}_{bcd} = \mathcal{L}_X R_{ab} = \mathcal{L}_X C^a{}_{bcd} = 0 \tag{3}$$

where R_{ab} ($\equiv R^c{}_{acb}$) and $C^a{}_{bcd}$ stand, respectively, for the components of the Ricci and the Conformal Weyl tensor. Also, from the Einstein's Field equations (EFE), it follows

$$\mathcal{L}_X T_{ab} = 0 \tag{4}$$

where T_{ab} is the energy momentum tensor representing the material content of the spacetime (M, g). It can easily be shown that whenever a proper HVF exists in a Lie algebra of HVF's $\mathcal{H}_r$, this necessarily contains an $(r-1)$-dimensional Lie subalgebra of KV $\mathcal{G}_{r-1}$; therefore one can always choose a basis for $\mathcal{H}_r$ in such a way that it contains at most one proper HVF, the $r-1$ remaining ones thus being KV's. If these vector fields in the basis of $\mathcal{H}_r$ are all complete vector fields, then $\mathcal{H}_r$ gives rise in a well known way to a Lie group of homotheties; otherwise, it gives rise to a local group of local homothetic transformations of M and, although the usual concepts of isotropy and orbits still hold, a little more care is required. We shall not go into the details here, but refer the reader to [2] for further information on this particular issue. It is also immediate to see from (1) that the Lie bracket of a proper HVF and a KV is a KV. Further information on the isotropy structure as well as on the fixed point structure of H_r can be found in [2, 6] (see also [9]).

In this paper we shall be concerned with spacetimes admitting a 3-parameter group of homotheties acting on non-null orbits, providing a classification of all possible Lie algebra structures (in terms of the Bianchi type of $\mathcal{H}_3$), and giving in each case the form of the metric as well as that of the proper HVF and the two KV's, in terms of local coordinates. We shall also present a few selected examples of spacetimes, satisfying the above properties, which can represent perfect fluid cosmological models, as well as vacuum solutions.

The paper is organized as follows: Section 2 contains a brief summary of results on groups of homotheties and spacetimes admitting them. In section 3 we present the Bianchi

classification of Lie algebras $\mathcal{H}_3$ along with some remarks on the topology of the orbits of the corresponding Lie subalgebras $\mathcal{G}_2$. The general form of the metric is provided in each case. Finally, section 4, contains a few selected examples of perfect fluid spacetimes admitting a maximal H_3; some of them we believe are new and, whenever this is possible, we relate our results to those already existing in the literature.

2 Preliminary Results

In this section we provide (without proof) some general results regarding spacetimes admitting Lie groups of HVF. In what is to follow we shall be assuming that a proper HVF exists; r will then denote the dimension of the Lie algebra of HVF $\mathcal{H}_r$ and $\mathcal{G}_{r-1}$ will be its associated Killing subalgebra. Most of the following results, together with their proofs, can be found in ref. [2].

1. The orbits of $\mathcal{H}_r$ and $\mathcal{G}_{r-1}$ can only coincide if they are 4-dimensional or 3-dimensional and null. Thus, the case in which we will be interested mainly ($r = 3$, non-null orbits) corresponds to a transitive action of the homothety group (and therefore, the dimensions of the orbits of $\mathcal{H}_3$ and $\mathcal{G}_2$ will be 3 and 2 respectively).

2. If $r = 11$ then M is flat. The cases $r = 10$, 9 are impossible, as it follows from consideration of the dimension of $\mathcal{G}_{r-1}$. $r = 8$ corresponds to M being a conformally flat, homogeneous generalized plane wave [1]. The case $r = 7$ implies that M is a type N, homogeneous or a conformally flat non-homogeneous generalized plane wave or one of the special Robertson-Walker spacetimes (or their equivalent, with Segre type $\{(1,11)1\}$ for the Ricci tensor). $r = 6$ implies that M is a type N, non-homogeneous, generalized plane wave. In the $r = 5$ case, the associated $\mathcal{G}_4$ subalgebra has necessarily 3-dimensional non-null orbits, the Petrov type being D, N or O for timelike Killing orbits, and D or O for spacelike ones.

3. If $r = 4$ and a multiply transitive action is assumed, then $\mathcal{H}_4$ and $\mathcal{G}_3$ have respectively 3-dimensional and 2-dimensional orbits.

4. Spacetimes admitting a 4-parameter group of homotheties acting transitively on M were studied by Rosquist and Jantzen [10] (see also [11] where a thorough study of vacuum Bianchi I solutions admitting a proper HVF is carried out).

5. The case $r = 3$ has an associated Killing subalgebra $\mathcal{G}_2$ and the respective dimensions of their orbits are 3 and 2 (see remark (1) above). In this case one can classify the Lie algebras $\mathcal{H}_3$ according to their Bianchi type (see for example [12]); the only possible types being those corresponding to soluble groups (I to VII in the previous reference), as it follows from the fact that $\mathcal{H}_3$ must contain a 2-dimensional subalgebra $\mathcal{G}_2$, which in all cases but one, turns out to be abelian. In this case (abelian $\mathcal{G}_2$), there are only two different topologies possible for the (non-null) orbits V_2; namely: V_2 diffeomorphic to $\mathcal{R}^2$, and V_2 diffeomorphic to $\mathcal{S}^1 \times \mathcal{R}$; and it follows that in the latter case [13] the only Bianchi type possible for $\mathcal{H}_3$ is I; as for the case $V_2 \cong \mathcal{R}^2$, all seven types can, in principle, occur.

3 Bianchi types of $\mathcal{H}_3$

The purpose of this section is to analyze the possible Bianchi types of $\mathcal{H}_3$, giving in each case the coordinate forms of the proper HVF and the metric tensor. We shall restrict ourselves to the case of non-null orbits, and furthermore we shall assume (as is customary) that the Killing orbits V_2 admit orthogonal 2-surfaces. We shall denote the KV's spanning $\mathcal{G}_2$ by ξ and η, and the proper HVF in the basis of $\mathcal{H}_3$ as X; also we shall treat separately the case where $\mathcal{G}_2$ is abelian from that where $\mathcal{G}_2$ is non-abelian.

3.1 Case $\mathcal{G}_2$ abelian

Under the above assumptions, the possible Bianchi types of $\mathcal{H}_3$ containing an abelian $\mathcal{G}_2$ are [12]:

$$
\begin{array}{rl}
(I) & [\xi, \eta] = [\xi, X] = [\eta, X] = 0 \\
(II) & [\xi, \eta] = [\xi, X] = 0 \quad [\eta, X] = \xi \\
(III) & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = 0 \\
(IV) & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = \xi + \eta \\
(V) & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = \eta \\
(VI) & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = q\eta \\
(VII) & [\xi, \eta] = 0 \quad [\xi, X] = \eta \quad [\eta, X] = -\xi + q\eta \quad (q^2 < 4)
\end{array}
$$

Assume now that the Killing orbits V_2 are spacelike and diffeomorphic to $\mathcal{R}^2$; since ξ and η commute, we locally have

$$\xi = \frac{\partial}{\partial x} \qquad \eta = \frac{\partial}{\partial y} \tag{5}$$

taking now two more coordinates, t and z, it follows from the assumption that the Killing orbits V_2 admit orthogonal surfaces, that the line element associated to the metric g can be written as

$$ds^2 = \Psi^{-2}\{-dt^2 + dz^2 + s^2 dy^2 + b^2(Pdy + dx)^2\} \tag{6}$$

where Ψ, s, b and P are all functions of t and z alone, their functional dependence on these coordinates to be determined (to some extent) by the HVF X in each case $(I - VII)$.

For timelike Killing orbits, also diffeomorphic to $\mathcal{R}^2$, one (locally) has:

$$\xi = \frac{\partial}{\partial t} \qquad \eta = \frac{\partial}{\partial z} \tag{7}$$

and the metric would then read:

$$ds^2 = \Psi^{-2}\{dx^2 + dy^2 + s^2dz^2 - b^2(dt + Pdz)^2\} \tag{8}$$

where Ψ, s, b and P are now functions of x and y, coordinates on the surfaces orthogonal to the Killing orbits.

The case $V_2 \cong \mathcal{S}^1 \times \mathcal{R}$, either spacelike (cylindrical symmetry) or timelike (stationary axially symmetric metric) is easily obtained from (5) and (6) (respectively (7) and (8)) by simply changing y (respectively z) to φ, angular coordinate (and then one has to impose the regularity condition on the axis, ref.[1] p.192; in order to ensure that φ has the standard periodicity 2π). It is precisely the fact that the axis of rotation is an (invariant) submanifold of the spacetime manifold, what implies that -if no other CKV exists on M- then $\mathcal{H}_3$ must be abelian (Bianchi type I) [13].

In what is to follow, and for the sake of simplicity, we shall assume that the Killing orbits V_2 are spacelike and diffeomorphic to $\mathcal{R}^2$, so that the forms (5) and (6) will hold for the KV's and the line element respectively. The case of timelike Killing orbits can be formally obtained by changing (x, y) into (t, z) in the expression (5) for the KV's ξ and η; and

$$t \longrightarrow ix \qquad y \longrightarrow z \qquad P \longrightarrow iP \tag{9}$$

in (6), to get the line element in this case.

Now, assume X is a proper-HVF satisfying (1) (with $\lambda = 1$); in a coordinate chart it will have an expression of the form

$$X = X^a(x^b)\partial_a \tag{10}$$

Specializing now the equation (1) to the HVF (10) and the metric (6) one sees that, assuming non-null homothetic orbits V_3, it is always possible to perform a coordinate change in the 2-spaces orthogonal to the Killing orbits; such that preserves the form of the metric and

brings the HVF (10) to one of the two following forms:

$$X = \partial_t + X^x(x,y)\partial_x + X^y(x,y)\partial_y \tag{11}$$

or

$$X = \partial_z + X^x(x,y)\partial_x + X^y(x,y)\partial_y \tag{12}$$

Assuming the form (11) for the proper HVF X (i.e.: 3-dimensional timelike homothetic orbits), equation (1), along with each particular Lie algebra structure, yields for every different Bianchi type (I) to (VII) the following forms for X and the functions Ψ, s, b and P appearing in (6),

$$(I) \qquad X = \partial_t \qquad \Psi = e^{-t}f(z) \quad s = \hat{s}(z) \quad b = \hat{b}(z) \quad P = \hat{p}(z) \tag{13}$$

$$(II) \qquad X = \partial_t + y\partial_x \qquad \Psi = e^{-t}f(z) \quad s = \hat{s}(z) \quad b = \hat{b}(z) \quad P = \hat{p}(z) - t \tag{14}$$

$$(III) \qquad X = \partial_t + x\partial_x \qquad \Psi = e^{-t}f(z) \quad s = \hat{s}(z) \quad b = e^{-t}\hat{b}(z) \quad P = e^{t}\hat{p}(z) \tag{15}$$

$$(IV) \qquad X = \partial_t + (x+y)\partial_x + y\partial_y \qquad \Psi = e^{-t}f(z) \tag{16}$$

$$s = e^{-t}\hat{s}(z) \quad b = e^{-t}\hat{b}(z) \quad P = \hat{p}(z) - t$$

$$(V) \qquad X = \partial_t + x\partial_x + y\partial_y \qquad \Psi = e^{-t}f(z) \tag{17}$$

$$s = e^{-t}\hat{s}(z) \quad b = e^{-t}\hat{b}(z) \quad P = \hat{p}(z)$$

$$(VI) \qquad X = \partial_t + x\partial_x + qy\partial_y \quad (q \neq 0, 1) \qquad \Psi = e^{-t}f(z) \tag{18}$$

$$s = e^{-qt}\hat{s}(z) \quad b = e^{-t}\hat{b}(z) \quad P = e^{(1-q)t}\hat{p}(z)$$

$$(VII) \qquad X = \partial_t - y\partial_x + (x+qy)\partial_y \quad (q^2 < 4) \qquad \Psi = e^{-t}f(z) \tag{19}$$

$$s = \frac{e^{-\frac{q}{2}t}\frac{\sqrt{4-q^2}}{2}a(z)}{(\sqrt{a(z)^2 + c(z)^2 + g(z)^2} + c(z)\cos(\sqrt{4-q^2}t) + g(z)\sin(\sqrt{4-q^2}t))^{\frac{1}{2}}}$$

$$b = e^{-\frac{q}{2}t}(\sqrt{a(z)^2 + c(z)^2 + g(z)^2} + c(z)\cos(\sqrt{4-q^2}t) + g(z)\sin(\sqrt{4-q^2}t))^{\frac{1}{2}}$$

$$P = \frac{q}{2} + \frac{\sqrt{4-q^2}}{2}\frac{-g(z)\cos(\sqrt{4-q^2}t) + c(z)\sin(\sqrt{4-q^2}t)}{\sqrt{a(z)^2 + c(z)^2 + g(z)^2} + c(z)\cos(\sqrt{4-q^2}t) + g(z)\sin(\sqrt{4-q^2}t)}$$

If P =constant two mutually orthogonal KV's exist in $\mathcal{G}_2$ and one can always, by means of a linear change of coordinates in the Killing orbits V_2, bring P to zero. It is worth noticing that this is not possible for families (II) and (IV). Note that in case VII, P =constant implies the existence of a third KV tangent to the Killing orbits V_2, therefore these orbits are of constant curvature and the Lie algebra of HVF is 4-dimensional.

The other ansatz for the proper HVF X, (12) corresponds to the homothetic orbits being spacelike, and would yield similar results to those above but with the role of the coordinates t and z reversed.

As for the case $V_2 \cong \mathcal{S}^1 \times \mathcal{R}$, and from the previous remarks on this issue, it follows that the metric would be that given in (6) (or (8)) exchanging y (or z) for φ, and with Ψ, s, b and P being those given in (13) (or their equivalent under the substitution (9) in the case of timelike Killing orbits).

3.2 Case $\mathcal{G}_2$ non-abelian

The Lie algebra structure in this case is

$$[\xi, \eta] = \xi \quad [\xi, X] = [\eta, X] = 0 \tag{20}$$

Assuming that the theorem of Bilyalov [14] and Defrise-Carter [15] holds (for a precise statement of the conditions under which this happens, see [7]), there exists a (smooth) function $\sigma = \sigma(x^c)$ such that ξ, η and X span a 3-dimensional Lie algebra of KV's in a spacetime $(M, \hat{g})$ where $\hat{g} = e^{-2\sigma} g$; i.e.: $(M, \hat{g})$ is a Bianchi type III spacetime. One can now adapt coordinates to ξ, η and X in $(M, \hat{g})$ as follows:

$$\xi = \frac{\partial}{\partial x^1} \qquad \eta = A\frac{\partial}{\partial x^3} + B\frac{\partial}{\partial x^1} + C\frac{\partial}{\partial x^2} \qquad X = \frac{\partial}{\partial x^3} \tag{21}$$

where A, B and C are functions of x^α, $\alpha = 1, 2, 3$. The commutation relations (20) imply: $A = A(x^2)$, $B = x^1 + B_0(x^2)$ and $C = C(x^2)$; and one can then always carry out a change

of coordinates $(x^{\alpha}) \to (x^{\alpha'})$ so as to write ξ, η and X as:

$$\xi = \frac{\partial}{\partial x^{1'}} \qquad \eta = x^{1'}\frac{\partial}{\partial x^{1'}} + \frac{\partial}{\partial x^{2'}} \qquad X = \frac{\partial}{\partial x^{3'}} \tag{22}$$

dropping now the primes and choosing a new coordinate x^4; it follows that the metric $\hat{g}$ can be written as

$$\hat{g}_{ab} = \begin{pmatrix} e^{-2x^2}a_{11} & e^{-x^2}a_{12} & e^{-x^2}a_{13} & 0 \\ e^{-x^2}a_{12} & a_{22} & a_{23} & 0 \\ e^{-x^2}a_{13} & a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{23}$$

where $\epsilon = \pm 1$ and $a_{\alpha\beta} = a_{\alpha\beta}(x^4)$. It is now immediate to find out the general form for the metric g; for from $g = e^{2\sigma}\hat{g}$ along with $\mathcal{L}_X g = 2g$ it readily follows that $\sigma = x^3 + \sigma_0(x^4)$ (since $\mathcal{L}_\xi g = \mathcal{L}_\eta g = 0$); redefining now the coordinate x^4 one has

$$g_{ab} = e^{2x^3}\begin{pmatrix} e^{-2x^2}A_{11} & e^{-x^2}A_{12} & e^{-x^2}A_{13} & 0 \\ e^{-x^2}A_{12} & A_{22} & A_{23} & 0 \\ e^{-x^2}A_{13} & A_{23} & A_{33} & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{24}$$

where again $A_{\alpha\beta} = A_{\alpha\beta}(x^4)$ and $\epsilon = \pm 1$. The case $\epsilon = +1$ corresponds to the 3-dimensional homothetic orbits being timelike, whereas $\epsilon = -1$ corresponds to spacelike homothetic orbits.

4 Examples

The aim of this section is to provide a sample of physically significant spacetimes admitting an H_3 on non-null orbits as maximal group of similarity.

4.1 Perfect Fluid Spacetimes

A perfect fluid spacetime (M, g) satisfies the EFE's for an energy-momentum tensor of the form:

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} \tag{25}$$

where u^a is the velocity flow of the fluid ($u^a u_a = -1$), μ is a positive function representing the energy density as measured by an observer comoving with the fluid, and p represents the

pressure, usually satisfying an equation of state of the form $p = p(\mu)$ (barotropic equation of state; the fluid is then said to be isentropic; i.e.: zero density of entropy production). If a KV $\mathcal{X}$ exists in the spacetime, one has [1]:

$$\mathcal{L}_{\mathcal{X}} u_a = \mathcal{L}_{\mathcal{X}} \mu = \mathcal{L}_{\mathcal{X}} p = 0 \tag{26}$$

and the existence of a proper HVF X implies in turn [16, 17]:

$$\mathcal{L}_X u_a = u_a \qquad \mathcal{L}_X \mu = -2\mu \qquad p = (\gamma - 1)\mu \tag{27}$$

where $\gamma \in [1, 2]$ in order to comply with the energy conditions [1]. Henceforth, all of the examples we shall present will correspond to the case of spacelike Killing orbits diffeomorphic to $\mathcal{R}^2$ and the choice (11) for the proper HVF. Thus, in the adapted coordinate system set up in section 3.1 (see for example (6)) we shall have:

$$\mu = e^{-2t}\hat{\mu}(z) \qquad u_t = -e^t f^{-1} \cosh\alpha(z) \qquad u_z = e^t f^{-1} \sinh\alpha(z) \tag{28}$$

where $\alpha(z)$ is a function to be determined via the field equations. The components of the 4-velocity on the Killing orbits are zero ($u_x = u_y = 0$) also as a consequence of the field equations. As for the remaining cases (timelike Killing orbits, Killing orbits diffeomorphic to a cylinder and/or the ansatz (12) for the proper HVF X), see remarks in the previous section concerning this. Nevertheless, no correspondence will exist -in general- between solutions obtained in all those various cases and those we will present here (the only similarity being the general form of the metrics under the changes of coordinates suggested in section 3, as it was already pointed out there).

Going back to the current case, the generic form of the spacetime metric will be that given in (6) with the functions Ψ, s, b and P that appear in (13)-(19), depending on the particular Bianchi type we are interested in. Furthermore, the generic forms of the fluid 4-velocity and of the energy density will be those given by (28). From the expression of

the velocity of the fluid, it is easy to see that, in general, it is non-geodesic, expanding and shearing; its vorticity being zero since u^a is orthogonal to the Killing orbits.

At this point it is convenient to split up our study into two cases:

4.1.1 The fluid flow velocity is tangent to the homothetic orbits.

In this case, and since we chose the coordinates t, x and y adapted to the homothetic orbits; it follows that $u_z = 0$, and therefore the fluid is comoving (for this particular choice of coordinates). One then has

$$u_t = -e^t f^{-1}(z) \qquad \dot{u}_z = -\frac{f'}{f} \tag{29}$$

$$\theta = e^{-t} f(k+3) \tag{30}$$

where a dash indicates differentiation with respect to z, θ stands for the expansion of the fluid ($\theta \equiv u^a{}_{;a}$); k is defined as $sb = e^{kt}\hat{s}(z)\hat{b}(z)$, and the remaining components of the 4-velocity u_a and the acceleration $\dot{u}_a$ are zero.

From the contracted Bianchi identities it follows

$$\gamma = \frac{2}{k+3} \tag{31}$$

Now, from the classification of $\mathcal{H}_3$ into Bianchi types, we see that for families I and II one has $\gamma = \frac{2}{3}$. Such a value for γ lies out of the interval permitted ($\gamma \in [1,2]$); nevertheless, it is physically significant since matter becomes attractive for $\gamma > \frac{2}{3}$; therefore the value $\frac{2}{3}$ may be of interest in inflationary models [18]. For family III one has $k = -1$ and hence $\gamma = 1$; i.e.: $p = 0$ (dust). Families IV and V correspond to $k = -2$, i.e.: $\gamma = 2$; that is: $p = \mu$ (stiff matter). For family VI, $k = -(q+1)$ (with $q \neq 0, 1$) and thus $\gamma = 2/(2-q)$; which taking into account the permitted values of γ, implies that $q \in (0,1)$. For family VII, $k = -q$ (with $q^2 < 4$) and therefore $\gamma = 2/(3-q)$.

Furthermore, it is possible to see from the field equations, that the families (III) and (V) admit no solutions of this type with $\mu \neq 0$.

The case when $P = 0$ (i.e.: $\mathcal{G}_2$ contains two mutually orthogonal KV's) and u^a is tangent to the homothetic orbits, has been thoroughly studied by Wainwright and collaborators in a series of papers [3, 4, 5] dedicated to investigate the role of self-similarity in Cosmology. They interpret these self-similar models as asymptotic states (at late times) of more general inhomogeneous cosmological models; since they are precisely those corresponding to the equilibrium points of the EFE's, written as an autonomous system, for orthogonally transitive G_2 Cosmologies.

From our remarks above, it follows that their solutions must be of the type VI (type I is ruled out since $\gamma \notin [1, 2]$, types III and V cannot admit solutions with $\mu \neq 0$; and type VII together with $P = 0$ implies the existence of a further KV tangent to the Killing orbits V_2 and the metric would then admit a non-transitive group H_4 of homotheties).

Eardley [17] studied the case of 3-dimensional spacelike homothetic orbits.

4.1.2 The fluid flow is not tangent to the homothetic orbits (tilted case).

In this case $u_z \neq 0$ for our particular choice of coordinates, and consequently the expressions for the acceleration $\dot{u}_a$ and the expansion θ of the fluid become more complicated, as well as the field equations. In orthogonal transitive abelian G_2 models, it is possible, though, to perform a change of coordinates in the t, z plane so as to bring the 4-velocity of the fluid to a comoving form, preserving the diagonal form of the induced metric there [9], as a consequence, the field equations can be written in a much simpler form. Since most of the solutions of these characteristics appearing in the literature are given in those coordinates, we found convenient to translate our results (6) and (13-19) to them. Obviously, the form of the proper HVF X will change and the coordinates will no longer be adapted to the homothetic orbits; thus, we next give the equivalents of (6) and (13)-(19) in the new coordinates. Following [19]; the metric can now be written as:

$$ds^2 = -A^2 dt^2 + B^2 dz^2 + r\{f(dx + w dy)^2 + f^{-1} dy^2\} \tag{32}$$

where A, B, r, f and w are functions of t and z and the two (commuting) KV's are $\xi = \frac{\partial}{\partial x}$ and $\eta = \frac{\partial}{\partial y}$ (same as before). The 4-velocity of the fluid is now:

$$u = A^{-1}\frac{\partial}{\partial t}, \quad \text{or} \quad \text{equivalently} \quad u_a = (-A, 0, 0, 0) \tag{33}$$

we can now use the remaining coordinate freedom in the t, z plane ($t \longrightarrow m(t)$, $z \longrightarrow n(z)$) to bring the (non-null) proper HVF X satisfying $(I - VII)$ to either of the following three forms:

$$(i) \qquad X = \partial_t + X^x(x,y)\partial_x + X^y(x,y)\partial_y \tag{34}$$

$$(ii) \qquad X = \partial_z + X^x(x,y)\partial_x + X^y(x,y)\partial_y \tag{35}$$

$$(iii) \qquad X = \partial_t + \partial_z + X^x(x,y)\partial_x + X^y(x,y)\partial_y \tag{36}$$

$X^x(x,y)$ and $X^y(x,y)$ being linear functions of the coordinates x and y, to be determined for each particular algebraic Bianchi type I to VII. Notice that (i) corresponds to u being tangent to the homothetic orbits, and therefore it has been dealt with above. (ii) corresponds to the orbits of the homothety group being spacelike (and also $u^a X_a = 0$); and this is the case studied by Eardley [17] (in this case, and since X and u are mutually orthogonal it follows $\gamma = 2$; i.e.: $p = \mu$ stiff matter). Finally, (iii) is precisely the case we are currently interested in; namely u not tangent to the homothetic orbits.

Specializing now the equation (1) to the metric (32) and the HVF X given by (36), we get for the metric functions

$$A^2 = e^{t+z}\hat{A}^2(t-z) \qquad B^2 = e^{t+z}\hat{B}^2(t-z) \tag{37}$$

in all seven types; and:

$$(I) \qquad r = e^{t+z}\hat{r}(t-z) \qquad f = \hat{f}(t-z) \qquad w = \hat{w}(t-z) \tag{38}$$

$$(II) \qquad r = e^{t+z}\hat{r}(t-z) \qquad f = \hat{f}(t-z) \qquad w = \hat{w}(t-z) - \frac{t+z}{2} \tag{39}$$

$$(III) \qquad r = e^{\frac{t+z}{2}}\hat{r}(t-z) \qquad f = e^{-\frac{t+z}{2}}\hat{f}(t-z) \qquad w = e^{\frac{t+z}{2}}\hat{w}(t-z) \tag{40}$$

$$(IV) \qquad r=\hat{r}(t-z) \qquad f=\hat{f}(t-z) \qquad w=\hat{w}(t-z)-\frac{t+z}{2} \tag{41}$$

$$(V) \qquad r=\hat{r}(t-z) \qquad f=\hat{f}(t-z) \qquad w=\hat{w}(t-z) \tag{42}$$

$$(VI) \qquad r=e^{\frac{1-q}{2}(t+z)}\hat{r}(t-z) \quad f=e^{-\frac{1-q}{2}(t+z)}\hat{f}(t-z) \quad w=e^{\frac{1-q}{2}(t+z)}\hat{w}(t-z) \tag{43}$$

$$(VII) \qquad r=e^{\frac{2-q}{2}(t+z)}\sqrt{\frac{4-q^2}{4}}\hat{r}(t-z) \tag{44}$$

$$f=\frac{\sqrt{\hat{r}^2+\hat{b}^2+\hat{c}^2}+\hat{b}\cos(\sqrt{\frac{4-q^2}{4}}(t+z))+\hat{c}\sin(\sqrt{\frac{4-q^2}{4}}(t+z))}{\sqrt{\frac{4-q^2}{4}}\hat{r}(t-z)}$$

$$w=\frac{q}{2}+\sqrt{\frac{4-q^2}{4}}\frac{\hat{b}\sin(\sqrt{\frac{4-q^2}{4}}(t+z))-\hat{c}\cos(\sqrt{\frac{4-q^2}{4}}(t+z))}{\sqrt{\hat{r}^2+\hat{b}^2+\hat{c}^2}+\hat{b}\cos(\sqrt{\frac{4-q^2}{4}}(t+z))+\hat{c}\sin(\sqrt{\frac{4-q^2}{4}}(t+z))}$$

where $\hat{b}$ and $\hat{c}$ are both functions of $(t-z)$.

It is interesting to notice that all diagonal ($w=0$), perfect fluid solutions of the form (32) (i.e.: admitting an orthogonally transitive abelian G_2 with flat spacelike orbits) and such that the functions A, B, r and f are separable in the variables t and z are already known [20]. Note that the only Bianchi types which can contain diagonal metrics such that, for them, the maximal isometry group is the abelian G_2 generated by ξ and η, are the types I, III, V and VI (families II and IV do not contain diagonal metrics, and the diagonal, type VII case admits a further KV). For them, the metric functions are all of the form

$$F=e^{a(t+z)}\phi(t-z) \quad , \quad a=constant \tag{45}$$

and it is immediate to prove that F is separable in t and z if and only if ϕ is of the form:

$$\phi=Ce^{k(t-z)} \quad , \quad C,k=constants \tag{46}$$

We next present a few solutions which have been obtained for the Bianchi types I, III and V assuming $w=0$ (diagonal), but which are not separable in the above sense.

Type I

$$ds^2=\frac{e^{t+z}}{{f_o}^2|1-e^{-2(t-z)}|c^2}\{-e^{t-z}dt^2+e^{-(t-z)}dz^2\}+e^{2t}dx^2+e^{2z}dy^2 \tag{47}$$

$$\mu = \frac{c^2 {f_o}^2}{e^{2t}} |1 - e^{-2(t-z)}|^{c^2} \quad , \quad p = \mu$$

Type III

$$ds^2 = e^{t+z} k^2 |1 - e^{-(t-z)}|^\beta \{-e^{t-z} dt^2 + dz^2\} + e^{t+z} dx^2 + dy^2 \tag{48}$$

$$\mu = \frac{1 - \beta}{4k^2 e^{2t} |1 - e^{-(t-z)}|^\beta} \quad , \quad p = \mu$$

For $\beta = 0$ in the above solution, the fluid has geodesic flow ($\dot{u}_a = 0$) and the metric admits a further spacelike KV which is not tangent to the Killing orbits V_2; the solution being therefore a special type of spatially homogeneous Bianchi cosmological model. For $\beta \neq 0$ the fluid is non-geodesic and the metric admits no further KV's.

Type V

$$ds^2 = \frac{e^{t+z}}{{f_o}^2} \left\{ -dt^2 \frac{c^2 \varphi^2}{1 - c^2 \varphi^2} + \frac{dz^2}{1 - c^2 \varphi^2} \right\} + \varphi^2 dx^2 + dy^2 \tag{49}$$

$$\mu = \frac{1 - c^2 \varphi^2}{2c\varphi^2} \frac{{f_o}^2}{e^{t+z}} \quad , \quad p = \mu$$

where φ is a function of $t - z$ given implicitly by:

$$c(t - z) = \ln \varphi - \frac{c^2}{2} \varphi^2 \tag{50}$$

Notice that all these solutions have a stiff matter equation of state ($p = \mu$) and therefore can be derived from vacuum solutions (also admitting an abelian G_2) using a method proposed by Wainwright et al. [21]

4.2 Vacuum Solutions

From our previous developments (see (13) and (17)) it is possible to find all vacuum solutions corresponding to types I and V in our classification. Solving the vacuum field equations for them we get, respectively:

Type I

$$
\begin{aligned}
ds^2 &= e^{2t}\Big\{ - dt^2 + dz^2 + \frac{(e^{2z} - (\alpha^2 + c^2)e^{-2z})^2}{\alpha^2 e^{-2z} + (e^z - ce^{-z})^2} dy^2 + \\
&+ (\alpha^2 e^{-2z} + (e^z - ce^{-z})^2) \left(\frac{2\alpha}{\alpha^2 e^{-2z} + (e^z - ce^{-z})^2} dy + dx \right)^2 \Big\}
\end{aligned} \tag{51}
$$

where α, c and β are constants.

Type V

$$
ds^2 = e^{2t} e^{\alpha z} \{-dt^2 + dz^2\} + dy^2 + dx^2 \tag{52}
$$

where α is a constant. It is immediate to see that this is (locally) Minkowski spacetime and it can be brought to the standard form by means of the following coordinate change:

$$
\hat{t} = \frac{1}{2}\left(\frac{e^{(1+\alpha)(t+z)}}{1+\alpha} + \frac{e^{(1-\alpha)(t-z)}}{1-\alpha} \right) \quad , \quad \hat{z} = \frac{1}{2}\left(\frac{e^{(1+\alpha)(t+z)}}{1+\alpha} - \frac{e^{(1-\alpha)(t-z)}}{1-\alpha} \right) \tag{53}
$$

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